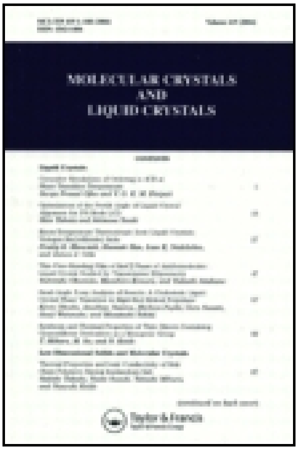

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Continuum Theory of Cholesteric Liquid Crystals

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Abstract—This paper discusses a continuum theory which has recently been proposed for liquid crystals of the cholesteric type. As in earlier continuum theories, the liquid crystal is regarded as an incompressible liquid with a preferred direction at each point, described by a unit vector. Solutions of the equations are investigated for shear flow between two flat plates, one at rest, and the second moving parallel to the first with a constant velocity. It is found that the theory predicts non-Newtonian behaviour, and that secondary flow and temperature variations occur. The predictions are compared with available experimental data, and one finds agreement in that a uniform apparent viscosity is predicted at large shear rates.

1. Introduction

The last ten years has seen increased activity in continuum theories of liquid crystals. This renewed interest began with Ericksen's work on anisotropic fluids,[1] which was soon followed by his hydrostatic theory of liquid crystals.[2] At about the same time, Ericksen[3] proposed conservation laws to describe the dynamical behaviour of liquid crystals. Later, on the basis of these conservation laws, Leslie[4] considered constitutive equations similar to those discussed by Ericksen in his theory of anisotropic fluids, and essentially reformulated that theory, which is thought to be inadequate to describe liquid crystals. Recently, however, Leslie[5] has considered a wider class of constitutive equations, and has proposed a theory for the nematic case, which is equivalent in equilibrium

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to Ericksen's hydrostatic theory. Using this theory, Leslie[5] and Ericksen[6] have made theoretical investigations related to the viscometry of nematic liquid crystals. Ericksen discusses this and other work in his paper in these Proceedings.[7] More recently, Leslie[8] has proposed a similar theory appropriate to the cholesteric mesophase. With it he offers an explanation of the spinning phenomenon noted by Lehmann.[9] An account of the above and other recent activity in continuum theories of liquid crystals may be obtained from the recent survey article by Ericksen.[10]

The aim of this paper is to consider the shear flow of cholesteric liquid crystals employing Leslie's theory, which is briefly outlined below. We consider a layer of liquid crystal confined between two parallel, plane plates, one of which is at rest and the other moving with constant speed along a straight line in its plane, and examine time independent solutions of the equations of motion in which the velocity, preferred direction, and temperature of the liquid may vary only with distance from the plates. In addition to the customary no slip condition, the boundary conditions considered are that the plates are maintained at the same constant temperature, and that the preferred direction aligns itself parallel to the plates such that they exert no couple stress on the liquid crystal. It is shown that the problem reduces to the solution of a pair of coupled, second order, non-linear differential equations, with two point boundary conditions, for the preferred direction. While we are unable to obtain explicit solutions of these equations, it is possible to partially integrate them to yield equations from which solutions may be readily computed. However, since little information is available at present concerning the various coefficients in the theory, no computation is attempted.

Assuming that solutions exist for the boundary conditions considered, their possible behaviour is discussed. At low rates of shear, the preferred direction appears to adopt twisted configurations, similar to those in the static theory. At high rates of shear, the indications are that, in certain circumstances, the flow dictates a uniform orientation throughout the liquid crystal, except in thin layers at the plates, in which the preferred direction changes to that required by the boundary conditions. From the equations, one can predict the order of magnitude of the shear stress necessary to achieve this uniform orientation. In general the apparent viscosity varies with rate of shear, and secondary flow and thermal variations occur.

In their experiments with cholesteric liquid crystals, Porter and Johnson[11] found marked non-Newtonian behaviour, the apparent viscosity decreasing with increasing shear rate to a uniform value at large shear rates. While a close comparison of theory with experiment is not possible at this stage, it would appear that there need be no conflict. Indeed, the theoretical estimate of the shear stress necessary to produce uniform orientation compares favourably with that required to produce a uniform value for the apparent viscosity in the experiments.

2. The Theory

The motion of the liquid crystal is referred to a fixed set of right-handed, Cartesian axes, and Cartesian tensor notation employed. As is usual in continuum theories of liquid crystals, the preferred direction is described by a unit vector d_i, and d_i and $-d_i$ are regarded as equivalent. We frequently refer to d_i as a director, and define a director velocity by

$$w_i = \frac{d}{dt} d_i$$

where d/dt denotes the material time derivative.

To describe the dynamical behaviour of the liquid crystal, we adopt the following conservation laws, which, apart from notational differences, are those proposed by Ericksen.[3] The liquid crystal is assumed to be incompressible, and therefore the velocity vector v_i must satisfy

$$v_{i,i} = 0 \tag{2.1}$$

The equation of linear momentum is

$$\rho \frac{dv_i}{dt} = \rho F_i + \sigma_{ji,j} \tag{2.2}$$

where ρ is the constant density, F_i body forces per unit mass, and σ_{ji} the stress tensor. In the absence of external forces such as magnetic fields influencing the preferred direction, the vector d_i is determined from

$$\rho_1 \frac{d}{dt} w_i = g_i + \pi_{ji,j} \tag{2.3}$$

where ρ_1 is a constant, g_i the intrinsic director body force, and π_{ji} the director stress tensor. If t_i and s_i are respectively the force per unit area and director force per unit area on a surface with unit normal n_i, we have

$$t_i = \sigma_{ji} n_j, \qquad s_i = \pi_{ji} n_j \tag{2.4}$$

The theory incorporates a couple stress τ_{ji}, given by

$$\tau_{ji} = e_{ipq} d_p \pi_{jq} \tag{2.5}$$

and as a result the equation of moment of momentum takes the form

$$\sigma_{ji} - \pi_{kj} d_{i,k} + g_j d_i = \sigma_{ij} - \pi_{ki} d_{j,k} + g_i d_j \tag{2.6}$$

Finally in the absence of external heat supply, the energy equation is

$$\rho \frac{dU}{dt} = \sigma_{ji} v_{i,j} + \pi_{ji} w_{i,j} - g_i w_i - q_{i,i} \tag{2.7}$$

where U is the internal energy per unit mass, and q_i the heat flux vector.

By the use of an entropy production inequality, Leslie[8] has proposed the following constitutive equations for cholesteric liquid crystals,

$$\begin{aligned}
\sigma_{ji} &= -p\,\delta_{ij} - \rho \frac{\partial F}{\partial d_{k,j}} d_{k,i} + \alpha e_{jkp}(d_p d_i)_{,k} + \hat{\sigma}_{ji}, \\
\pi_{ji} &= \beta_j d_i + \rho \frac{\partial F}{\partial d_{i,j}} + \alpha e_{ijk} d_k, \\
g_i &= \gamma d_i - (\beta_j d_i)_{,j} - \rho \frac{\partial F}{\partial d_i} - \alpha e_{ijk} d_{k,j} + \hat{g}_i
\end{aligned} \tag{2.8}$$

where p and γ are arbitrary scalars, and β_i an arbitrary vector, arising from the constraints of incompressibility and the director having fixed magnitude; F is the free energy, and Leslie adopted the expression given by Frank[12] for this quantity,

$$2\rho F = 2\rho k_0 + 2k_2 d_i e_{ijk} d_{k,j} + (k_{11} - k_{22} - k_{24}) d_{i,i} d_{j,j} + k_{22} d_{i,j} d_{i,j} + k_{24} d_{i,j} d_{j,i} + (k_{33} - k_{22}) d_i d_j d_{k,i} d_{k,j} \tag{2.9}$$

The internal energy U and the entropy S are related to the free energy by

$$U = F + TS, \qquad S = -\frac{\partial F}{\partial T} \tag{2.10}$$

where the scalar T is temperature. The quantities $\hat{\sigma}_{ji}$ and $\hat{g}_i$ are the non-equilibrium contributions to the above equations, and these and the heat flux vector are given by

$$\begin{aligned} \hat{\sigma}_{ji} &= \mu_1 d_k d_p A_{kp} d_i d_j + \mu_2 d_j N_i + \mu_3 d_i N_j + \mu_4 A_{ij} \\ &\quad + \mu_5 d_j d_k A_{ki} + \mu_6 d_i d_k A_{kj} + \mu_7 d_j e_{ipq} d_p T_{,q} + \mu_8 d_i e_{jpq} d_p T_{,q}, \\ \hat{g}_i &= \lambda_1 N_i + \lambda_2 d_j A_{ji} + \lambda_3 e_{ijk} d_j T_{,k}, \\ q_i &= \kappa_1 T_{,i} + \kappa_2 d_j T_{,j} d_i + \kappa_3 e_{ijk} d_j N_k + \kappa_4 e_{ijk} d_j A_{kp} d_p \end{aligned} \tag{2.11}$$

where

$$A_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}), \qquad N_i = w_i + \tfrac{1}{2}(v_{j,i} - v_{i,j}) d_j \tag{2.12}$$

and

$$\lambda_1 = \mu_2 - \mu_3, \quad \lambda_2 = \mu_5 - \mu_6, \quad \lambda_3 = \mu_7 - \mu_8 \tag{2.13}$$

The Eqs. (2.13) arise since Eq. (2.6) is treated as an identity to avoid the number of equations exceeding the number of unknowns. In general the coefficients appearing in Eqs. (2.8), (2.9) and (2.11) are functions of temperature. However, for the present investigation we regard them as constants. Finally, the coefficients occurring in Eqs. (2.11) must satisfy the inequality

$$\hat{\sigma}_{ji} A_{ij} - \hat{g}_i N_i - q_i T_{,i}/T - T \frac{\partial}{\partial T}(\phi_i/T) T_{,i} \geqslant 0 \tag{2.14}$$

where

$$\phi_i = \alpha e_{ijk} d_j (N_k + A_{kp} d_p) \tag{2.15}$$

for all deformations.

The static terms in the constitutive Eqs. (2.8) differ from those given by Ericksen[2] by the introduction of the terms involving the coefficient α and the vector β_i. Since the latter terms do not appear in the equations of motion, nor in the boundary conditions, they need not concern us further. The terms with coefficient α arise in a natural way by the use made of the entropy production inequality proposed by Müller.[13] However, the need to include them in a theory of liquid crystals has yet to be established.

3. Shear Flow

We consider a layer of liquid crystal which is sheared between two parallel, infinite flat plates, one of which is at rest, and the other moving with constant speed V along a straight line in its own plane. A set of right-handed Cartesian co-ordinates (x, y, z) is chosen so that the plates occupy the planes $z = h$ and $z = -h$, where h is a constant, and such that the former plate moves parallel to the positive x-axis. It is natural to examine solutions of the above equations in which

$$v_x = u(z), \qquad v_y = v(z), \qquad v_z = 0,$$

$$d_x = \cos\theta(z)\cos\phi(z), \quad d_y = \cos\theta(z)\sin\phi(z), \quad d_z = \sin\theta(z), \quad (3.1)$$

$$T = f(z)$$

the unknown functions depending only upon the co-ordinate z.

The Eq. (2.1) is satisfied automatically by the above choice for the velocity vector. Attention is confined to situations in which the body forces F_i are conservative, so that

$$\rho F_i = -\chi_{,i}$$

χ being the appropriate scalar potential. Consequently, Eqs. (2.2) reduce to

$$\frac{d}{dz}\sigma_{zx} - \frac{\partial}{\partial x}(p + \chi) = 0, \; \frac{d}{dz}\sigma_{zy} - \frac{\partial}{\partial y}(p + \chi) = 0, \; \frac{\partial}{\partial z}(\sigma_{zz} - \chi) = 0 \quad (3.2)$$

These have the general solution

$$\sigma_{zx} = a + a_1 z, \qquad \sigma_{zy} = b + b_1 z, \qquad \sigma_{zz} = \chi - p_0 - a_1 x - b_1 y \tag{3.3}$$

where a, a_1, b, b_1 and p_0 are arbitrary constants. For the present we discuss the case in which a_1 and b_1 are equal to zero. With the above constitutive assumptions, it follows that

$$\begin{aligned} \{H_1(\theta) + \cos^2\phi H_2(\theta)\}\xi + H_2(\theta)\cos\phi\sin\phi\eta + H_3(\theta)\sin\phi\zeta &= a, \\ \{H_1(\theta) + \sin^2\phi H_2(\theta)\}\eta + H_2(\theta)\cos\phi\sin\phi\xi - H_3(\theta)\cos\phi\zeta &= b \end{aligned} \tag{3.4}$$

where

$$2\xi = \frac{du}{dz}, \qquad 2\eta = \frac{dv}{dz}, \qquad \zeta = \frac{df}{dz} \tag{3.5}$$

and

$$\begin{aligned} H_1(\theta) &= \mu_4 + (\mu_5 - \mu_2)\sin^2\theta, \\ H_2(\theta) &= (2\mu_1\sin^2\theta + \mu_3 + \mu_6)\cos^2\theta, \\ H_3(\theta) &= \mu_7\sin\theta\cos\theta \end{aligned} \tag{3.6}$$

Straightforward manipulation yields from Eqs. (3.4)

$$\{H_1(\theta) + H_2(\theta)\}(\xi\cos\phi + \eta\sin\phi) = a\cos\phi + b\sin\phi, \tag{3.7}$$
$$H_1(\theta)(\xi\sin\phi - \eta\cos\phi) + H_3(\theta)\zeta = a\sin\phi - b\cos\phi$$

For the flow considered, the energy Eq. (2.7) reduces to

$$2(\hat{\sigma}_{zx}\xi + \hat{\sigma}_{zy}\eta) - \frac{d}{dz}q_z = 0 \tag{3.8}$$

In such problems, non-linear dissipative terms in the energy equation are generally regarded as negligible. Since it does not seem possible to include all effects, we adopt this approximation, and ignore the first two terms in Eq. (3.8). Recalling the constitutive assumption (2.11), one obtains

$$K_1(\theta)\zeta + K_2(\theta)(\xi\sin\phi - \eta\cos\phi) = c \tag{3.9}$$

where

$$K_1(\theta) = \kappa_1 + \kappa_2\sin^2\theta, \qquad K_2(\theta) = (\kappa_3 - \kappa_4)\sin\theta\cos\theta \tag{3.10}$$

and c is an arbitrary constant. Eqs. (3.7) and (3.9) yield three equations which may be solved to obtain expressions for the first derivatives ξ, η and ζ in terms of the angles θ and ϕ.

With the assumptions (3.1), Eqs. (2.3) become

$$\frac{d}{dz}\pi'_{zx} + g'_x = 0, \qquad \frac{d}{dz}\pi'_{zy} + g'_y = 0, \qquad \frac{d}{dz}\pi'_{zz} + g'_z = 0 \tag{3.11}$$

where

$$\pi'_{ji} = \pi_{ji} - \beta_j d_i, \qquad g'_i = g_i + (\beta_j d_i)_{,j}$$

After some manipulation, these equations reduce to

$$F_1(\theta)\frac{d^2\theta}{dz^2} + \tfrac{1}{2}\frac{d}{d\theta}F_1(\theta)\left(\frac{d\theta}{dz}\right)^2 - \tfrac{1}{2}\frac{d}{d\theta}F_2(\theta)\left(\frac{d\phi}{dz}\right)^2 \tag{3.12}$$

$$- 2k_2 \sin\theta\cos\theta\frac{d\phi}{dz} + (\lambda_1 + \lambda_2\cos 2\theta)(\xi\cos\phi + \eta\sin\phi) = 0$$

and

$$F_2(\theta)\frac{d^2\phi}{dz^2} + \frac{d}{d\theta}F_2(\theta)\frac{d\theta}{dz}\frac{d\phi}{dz} + 2k_2\sin\theta\cos\theta\frac{d\theta}{dz} \tag{3.13}$$

$$+ (\lambda_1 - \lambda_2)\sin\theta\cos\theta(\xi\sin\phi - \eta\cos\phi) - \lambda_3\zeta\cos^2\theta = 0$$

where

$$F_1(\theta) = k_{11}\cos^2\theta + k_{33}\sin^2\theta, \tag{3.14}$$

$$F_2(\theta) = (k_{22}\cos^2\theta + k_{33}\sin^2\theta)\cos^2\theta$$

The no slip condition is assumed to hold, and the plates are maintained at the same constant temperature. Therefore the boundary conditions for velocity and temperature are

$$u(h) = V, \quad u(-h) = v(h) = v(-h) = 0, \quad T(h) = T(-h) = T_0 \tag{3.15}$$

where T_0 is a constant. The boundary conditions to be applied to the preferred direction are less obvious. For the present we assume that the orientation aligns itself parallel to the plates, and such that they apply zero torque about the z-axis to the liquid crystal. The relevant component of the couple stress is

$$\tau_{zz} = F_2(\theta)\frac{d\phi}{dz} - (k_2 - \alpha)\cos^2\theta$$

Consequently the boundary conditions for θ and ϕ are

$$(h) = \theta(-h) = 0, \qquad \frac{d\phi(h)}{dz} = \frac{d\phi(-h)}{dz} = (k_2 - \alpha)/k_{22} \tag{3.16}$$

This choice yields twisted structures in static configurations, similar to those observed in cholesteric liquid crystals, and, as Leslie[8] has shown, it also offers an explanation of the spinning phenomenon noted by Lehmann.[9]

The symmetry of the problem leads one to examine solutions in which the functions θ, v and f are even functions of z, and ϕ and $u - V/2$ are odd functions. This is readily achieved by setting the constants b and c equal to zero, since one then obtains from Eqs. (3.7) and (3.9)

$$\begin{aligned} \xi &= a\left\{1 + \sin^2\phi\left(\frac{H_2(\theta)K_1(\theta) + H_3(\theta)K_2(\theta)}{H_1(\theta)K_1(\theta) - H_3(\theta)K_2(\theta)}\right)\right\}\Big/(H_1(\theta) + H_2(\theta)), \\ \eta &= -\frac{a\sin\phi\cos\phi(H_2(\theta)K_1(\theta) + H_3(\theta)K_2(\theta))}{(H_1(\theta) + H_2(\theta))(H_1(\theta)K_1(\theta) - H_3(\theta)K_2(\theta))}, \\ \zeta &= -a\sin\phi K_2(\theta)/(H_1(\theta)K_1(\theta) - H_3(\theta)K_2(\theta)) \end{aligned} \tag{3.17}$$

and the solutions have the required property provided θ is even and ϕ odd. The boundary conditions (3.15) are satisfied by an appropriate choice of the constant a. It remains to solve Eqs. (3.12) and (3.13) which reduce to

$$F_1(\theta)\frac{d^2\theta}{dz^2} + \tfrac{1}{2}\frac{d}{d\theta}F_1(\theta)\left(\frac{d\theta}{dz}\right)^2 - \tfrac{1}{2}\frac{d}{d\theta}F_2(\theta)\left(\frac{d\phi}{dz}\right)^2 - 2k_2\sin\theta\cos\theta\frac{d\phi}{dz} + a\cos\phi Q(\theta) = 0 \tag{3.18}$$

and

$$F_2(\theta)\frac{d^2\phi}{dz^2} + \frac{d}{d\theta}F_2(\theta)\frac{d\theta}{dz}\frac{d\phi}{dz} + 2k_2\sin\theta\cos\theta\frac{d\theta}{dz} - a\sin\phi P(\theta) = 0 \tag{3.19}$$

where

$$\begin{aligned} Q(\theta) &= (\lambda_1 + \lambda_2\cos 2\theta)/(H_1(\theta) + H_2(\theta)), \\ P(\theta) &= \frac{(\lambda_2 - \lambda_1)\sin\theta\cos\theta K_1(\theta) - \lambda_3\cos^2\theta K_2(\theta)}{H_1(\theta)K_1(\theta) - H_3(\theta)K_2(\theta)} \end{aligned} \tag{3.20}$$

subject to the boundary conditions

$$\phi(0) = 0, \qquad \frac{d}{dz}\theta(0) = 0, \qquad \theta(h) = 0, \qquad \frac{d}{dz}\phi(h) = (k_2 - \alpha)/k_{22} \tag{3.21}$$

Eq. (3.19) may at once be written

$$\frac{d}{dz}\left(F_2(\theta)\frac{d\phi}{dz} - k_2 \cos^2 \theta\right) = a \sin \phi P(\theta) \tag{3.22}$$

Also, if one multiplies Eq. (3.18) by $d\theta/dz$, and Eq. (3.19) by $d\phi/dz$, and adds, one obtains

$$\frac{d}{dz}\left\{F_1(\theta)\left(\frac{d\theta}{dz}\right)^2 + F_2(\theta)\left(\frac{d\phi}{dz}\right)^2\right\} = 2a\left(\sin \phi P(\theta)\frac{d\phi}{dz} - \cos \phi Q(\theta)\frac{d\theta}{dz}\right) \tag{3.23}$$

Eqs. (3.22) and (3.23) promptly yield

$$\begin{aligned} F_2(\theta)\frac{d\phi}{dz} &= F_2(\theta_1)\beta_1 - k_2 \cos^2 \theta_1 + k_2 \cos^2 \theta + a\int_0^z \sin \phi P(\theta)\, dz, \\ F_1(\theta)\left(\frac{d\theta}{dz}\right)^2 &+ F_2(\theta)\left(\frac{d\phi}{dz}\right)^2 \\ &= F_2(\theta_1)\beta_1^2 + 2a\left(\int_0^\phi \sin \phi P(\theta)\, d\phi + \int_\theta^{\theta_1} \cos \phi\, Q(\theta)\, d\theta\right) \end{aligned} \tag{3.24}$$

where $\theta_1 = \theta(0)$ and $\beta_1 = d(\phi(0))/dz$. By assigning values to the constants θ_1 and β_1, one may perform a step by step integration of the Eqs. (3.24). Alternatively, one finds

$$\begin{aligned} F_2(\theta)\frac{d\phi}{dz} &= k_2 \cos^2 \theta - \alpha - a\int_z^h \sin \phi P(\theta)\, dz, \\ F_1(\theta)\left(\frac{d\theta}{dz}\right)^2 &+ F_2(\theta)\left(\frac{d\phi}{dz}\right)^2 \\ &= k_{11}\beta_2^2 + (k_2 - \alpha)^2/k_{22} - 2a\left(\int_0^\theta \cos \phi Q(\theta)\, d\theta + \int_\phi^{\phi_1} \sin \phi P(\theta) d\phi\right) \end{aligned} \tag{3.25}$$

where $\phi_1 = \phi(h)$ and $\beta_2 = d(\theta(h))/dz$. By an appropriate choice of the constants θ_1 and β_1, or ϕ_1 and β_2, one hopes to obtain solutions which satisfy the conditions (3.21). For the present we assume that this is possible, and discuss the behaviour of the resulting solutions. As a preliminary, however, some information concerning certain coefficients in the equations is collected together.

When the constant a is zero, a solution of Eqs. (3.25), subject to the conditions (3.21), is

$$\theta = 0, \qquad \phi = z/\tau, \qquad \tau = k_{22}/(k_2 - \alpha) \tag{3.26}$$

Static configurations of this type are commonly observed in cholesteric liquid crystals, and typical values for the parameter τ tend to be small, of the order of 10^{-4} cm. (compare the remarks of Frank[12] and Ericksen[14]). When the coefficient α is zero, it appears reasonable to propose that this commonly observed static solution be that with least energy, as remarked by Ericksen.[15] This leads to the inequalities

$$k_{11} \geqslant 0, \qquad k_{22} \geqslant 0, \qquad k_{33} \geqslant 0 \tag{3.27}$$

as may be seen for example from the expression given by Frank.[12] However, when α is non-zero, the inequalities (3.27) may again be motivated by the requirement that a perturbation from the solution (3.26) should increase the energy.

Employing Eq. $(3.24)_2$, or $(3.25)_2$, and the boundary conditions (3.21), it follows that

$$k_{11}\beta_2^2 + k_{22}\tau^{-2} = F_2(\theta_1)\beta_1^2 + 2a\left(\int_0^{\theta_1} \cos\phi Q(\theta)\, d\theta + \int_0^{\phi_1} \sin\phi P(\theta) d\phi\right) \tag{3.28}$$

As a consequence of the inequalities (3.27), the terms on the left-hand side of this equation are both positive. Therefore, when the constant a is small compared with the left-hand side of Eq. (3.28), it is necessary that β_1 be at least of the same order as τ^{-1}, assuming that the constants k_{22} and k_{33} are of the same order. From these considerations, it appears that the twist in the angle ϕ persists throughout the flow when the shear stress is small.

When the shear stress a is sufficiently large, the form of Eqs. (3.18) and (3.19) suggests a solution in which

$$\theta = \theta_0 = \tfrac{1}{2}\cos^{-1}(-\lambda_1/\lambda_2), \qquad \phi = 0 \tag{3.29}$$

in all of the flow, apart from thin layers at the plates in which the solution changes rapidly to meet the boundary conditions at $z = \pm h$. One necessary condition for a solution of this type is that

$$|\lambda_1| \leqslant |\lambda_2| \neq 0 \tag{3.30}$$

From the relationship (3.28), one sees that a second necessary condition is that the shear stress a be at least of the order of $k_{22}\tau^{-2}$, which is equivalent to the lengths τ, $(k_{22}/a)^{1/2}$ and $(k_2 - \alpha)/a$ all being of the same order of magnitude. Assuming that the three Frank constants k_{11}, k_{22} and k_{33} are of the same order, an inspection of the coefficients in Eqs. (3.18) and (3.19) leads one to add a third condition

$$h \gg \tau \tag{3.31}$$

since the length scales in the differential equations are all of order τ. Thus, when the conditions (3.30) and (3.31) are satisfied, and a is of the order $k_{22}\tau^{-2}$, or greater, it is not unreasonable to expect the solution (3.29) to hold throughout the channel, except in layers of thickness of order τ, or less, at the plates. In these "boundary layers", the orientation changes rapidly from that dictated by the flow to that required by the boundary conditions which have been imposed.

From Eqs. (3.17), one finds that

$$u = 2a\int_{-h}^{z}\left\{1 + \sin^2\phi\left(\frac{H_2(\theta)K_1(\theta) + H_3(\theta)K_2(\theta)}{H_1(\theta)K_1(\theta) - H_3(\theta)K_2(\theta)}\right)\right\} \div (H_1(\theta) + H_2(\theta))\,dz,$$

$$v = -2a\int_{-h}^{z}\frac{\sin\phi\cos\phi(H_2(\theta)K_1(\theta) + H_3(\theta)K_2(\theta))}{(H_1(\theta) + H_2(\theta))(H_1(\theta)K_1(\theta) - \mathrm{H}_3(\theta)K_2(\theta))}\,dz, \tag{3.32}$$

$$T = T_0 - a\int_{-h}^{z}\sin\phi K_2(\theta)/(H_1(\theta)K_1(\theta) - H_3(\theta)K_2(\theta))\,dz$$

In general, therefore, this theory predicts that secondary flow occurs when one shears cholesteric liquid crystals, and that the temperature varies throughout the layer. From the first of Eqs. (3.32), it follows that

$$V = 4a\int_0^h \left\{1 + \sin^2\phi\left(\frac{H_2(\theta)K_1(\theta) + H_3(\theta)K_2(\theta)}{H_1(\theta)K_1(\theta) - H_3(\theta)K_2(\theta)}\right)\right\} \div (H_1(\theta) + H_2(\theta))\,dz \qquad (3.33)$$

This equation provides rather an involved relationship between V, h and a. One can define an apparent viscosity μ in the usual manner,

$$\mu = 2ah/V \qquad (3.34)$$

Hence, from Eq. (3.33),

$$\mu = \left[\frac{2}{h}\int_0^h \left\{1 + \sin^2\phi\left(\frac{H_2(\theta)K_1(\theta) + H_3(\theta)K_2(\theta)}{H_1(\theta)K_1(\theta) - H_3(\theta)K_2(\theta)}\right)\right\} \div (H_1(\theta) + H_2(\theta))\,dz\right]^{-1} \qquad (3.35)$$

so that the apparent viscosity varies with the rate of shear and the gap width. When the conditions (3.30) and (3.31) hold, and the shear stress a is sufficiently large, one expects from equation (3.35) that the apparent viscosity μ approaches a limiting value

$$\bar{\mu} = \tfrac{1}{2}(H_1(\theta_0) + H_2(\theta_0)). \qquad (3.36)$$

In their experiments Porter and Johnson[11] found that the viscosity of cholesteric liquid crystals varied with the rate of shear, the degree of non-Newtonian behaviour increasing as the temperature decreased. In all cases, the viscosity tended to uniform values at high shear rates, and in most cases shear rates of the order of 10^4 sec^{-1}, or greater, were required to achieve these uniform values. The theory discussed in this paper indicates that the apparent viscosity has a constant value, given by equation (3.36), when the shear stress is of the order of $k_{22}\tau^{-2}$, or greater, and provided that the gap width is not too small. This seems to compare favourably with the data of Porter and Johnson, since, as mentioned earlier, the length τ is in general very small.

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